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# STABILITY OF DEFORMATION OF ISOTROPIC HYPERELASTIC BODIES

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*The equations relating stress rates to strain rates are formulated and conditions of stable deformation of isotropic hyperelastic bodies are described. Stress–strain relations are presented for pure shear and uniaxial and axisymmetric loading of a material with a constitutive function obtained by generalization of the constitutive function of Hooke's law. In the case of small strains, the diagrams virtually coincide with the linear diagrams following from Hooke's law. Ramification of solutions and transition to declining diagrams begin at the same time, irrespective of values of the constants of the material, when large stresses of the order of the shear modulus are reached.*

**1. Determination of Strains.** We consider two interrelated (Cartesian and curvilinear) coordinate systems in three-dimensional Euclidean space. We denote the Cartesian and curvilinear coordinates of material points in their initial positions at the initial moment $\tau = 0$ by $y^i$ and $x^i$, respectively, and those at the current moment $\tau$ by $\hat{y}^i$ and $\hat{x}^i$, respectively. The radius-vectors of points, basis vectors, and the metric tensor of the curvilinear coordinate system vary from $\boldsymbol{R} = y^i\boldsymbol{k}_i$, $\boldsymbol{l}_i = \boldsymbol{R}_{,x^i} = y^n_{,x^i}\,\boldsymbol{k}_n$, and $g_{ij} = \boldsymbol{l}_i \cdot \boldsymbol{l}_j$ at the moment $\tau = 0$ to $\hat{\boldsymbol{R}} = \hat{y}^i\boldsymbol{k}_i$, $\hat{\boldsymbol{l}}_i = \hat{\boldsymbol{R}}_{,\hat{x}^i} = \hat{y}^n_{,\hat{x}^i}\boldsymbol{k}_n$, and $\hat{g}_{ij} = \hat{\boldsymbol{l}}_i \cdot \hat{\boldsymbol{l}}_j$ at the moment $\tau$ ($\boldsymbol{k}_i$ are the basis vectors of the Cartesian coordinate system). The point displacement vectors are $\boldsymbol{u} = \hat{\boldsymbol{R}} - \boldsymbol{R} = w^i\boldsymbol{k}_i = u^i\boldsymbol{l}_i = \hat{u}^i\hat{\boldsymbol{l}}_i$, where $w^n = u^i y^n_{,x^i} = \hat{u}^i\hat{y}^n_{,\hat{x}^i}$. We write the basis vectors of the accompanying coordinate systems as $\hat{\boldsymbol{R}}_{,x^i} = \boldsymbol{l}_i + u^n_{,i}\boldsymbol{l}_n$ and $\boldsymbol{R}_{,\hat{x}^i} = \hat{\boldsymbol{l}}_i - \hat{u}^n_{;i}\hat{\boldsymbol{l}}_n$. Here and below the subscripts and superscripts $i$, $j$, $m$, and $n$ take the values 1, 2, and 3; summation from 1 to 3 is performed over repeated indices. The variables in the subscript after the comma denote partial differentiation; the subscript $i$ after the comma or semicolon denotes covariant differentiation with respect to $x^i$ or $\hat{x}^i$, respectively. Covariant differentiation with respect to $x^i$ and $\hat{x}^i$ is performed in the same curvilinear coordinate system, but the differentiated vectors and tensors are resolved into the different basis vectors $\boldsymbol{l}_i$ and $\hat{\boldsymbol{l}}_i$, respectively.

We introduce the Green and Almansi strain tensors $e = e_{ij}\boldsymbol{l}^i\boldsymbol{l}^j$ and $\hat{e} = \hat{e}_{ij}\hat{\boldsymbol{l}}^i\hat{\boldsymbol{l}}^j$, where

$$e_{ij} = (\hat{g}_{mn}\hat{x}^m_{,x^i}\hat{x}^n_{,x^j} - g_{ij})/2 = (u_{i,j} + u_{j,i} + u^n_{,i}\,u_{n,j})/2,$$
$$\hat{e}_{ij} = (\hat{g}_{ij} - g_{mn}x^m_{,\hat{x}^i}x^n_{,\hat{x}^j})/2 = (\hat{u}_{i;j} + \hat{u}_{j;i} - \hat{u}^n_{;i}\,\hat{u}_{n;j})/2, \qquad (1.1)$$

which represent strain at the same material point at the current moment and are related by the equalities $\hat{e}_{mn} = e_{ij}x^i_{,\hat{x}^m}x^j_{,\hat{x}^n}$ and $e_{ij} = \hat{e}_{mn}\hat{x}^m_{,x^i}\hat{x}^n_{,x^j}$. To determine the volume strain $\varepsilon_V = J - 1$, we use the Jacobian of transformation of the initial Cartesian coordinates to the current coordinates $J$ of the material points expressed in terms of the components of the strain tensors [1]

$$J = (\det G)^{-1/2}[\det (G + 2E)]^{1/2} = (\det \hat{G})^{1/2}[\det (\hat{G} - 2\hat{E})]^{-1/2}, \qquad (1.2)$$

where $E = \|e_{ij}\|$, $\hat{E} = \|\hat{e}_{ij}\|$, $G = \|g_{ij}\|$, and $\hat{G} = \|\hat{g}_{ij}\|$ are matrices consisting of the covariant components of the strain and metric tensors ($i$ and $j$ are the row and column numbers, respectively).

Strain rates are determined by two tensors $\eta = \eta_{ij}\boldsymbol{l}^i\boldsymbol{l}^j$ and $\hat{\eta} = \hat{\eta}_{ij}\hat{\boldsymbol{l}}^i\hat{\boldsymbol{l}}^j$ with the components $\eta_{ij} = (v_{i,j} + v_{j,i} + u^n_{,i}v_{n,j} + u^n_{,j}v_{n,i})/2$, $\hat{\eta}_{ij} = (\hat{v}_{i;j} + \hat{v}_{j;i})/2$, which are related by

$$\hat{\eta}_{mn} = \eta_{ij}x^i_{,\hat{x}^m}x^j_{,\hat{x}^n}, \qquad \eta_{ij} = \hat{\eta}_{mn}\hat{x}^m_{,x^i}\,\hat{x}^n_{,x^j}. \qquad (1.3)$$

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The tensor $\eta = \dot{e}$ is the rate of variation of the Green strain tensor, $\boldsymbol{v} = \dot{\boldsymbol{u}} = \dot{w}^i \boldsymbol{k}_i = v^i \boldsymbol{l}_i = \hat{v}^i \hat{\boldsymbol{l}}_i$, and $\dot{J} = J\hat{\eta}_i^i$ (the dot denotes differentiation with respect to time $\tau$). The motion of the neighborhood of a material point at every moment is combined from strain, translation with rate $\boldsymbol{v}$, and rigid-body rotation with angular velocity vector $\boldsymbol{\omega} = \omega^i \boldsymbol{k}_i$, where $\omega^m = [(\boldsymbol{v}_{,\hat{y}^n} \cdot \boldsymbol{k}_l) - (\boldsymbol{v}_{,\hat{y}^l} \cdot \boldsymbol{k}_n)]/2 = (\dot{w}^l_{,\hat{y}^n} - \dot{w}^n_{,\hat{y}^l})/2$ ($m, n, l$ is even rearrangement of the subscripts and superscripts 1, 2, and 3). The Cartesian components of the tensor $\hat{\eta} = \hat{\eta}_{ij}^{(d)} \boldsymbol{k}_i \boldsymbol{k}_j$ take the values of $\hat{\eta}_{ij}^{(d)} = [(\dot{\boldsymbol{v}}_{,\hat{y}^i} \cdot \boldsymbol{k}_j) + (\boldsymbol{v}_{,\hat{y}^j} \cdot \boldsymbol{k}_i)]/2 = (\dot{w}^j_{,\hat{y}^i} + \dot{w}^i_{,\hat{y}^j})/2$.

**2. Principal Extension Ratios.** The principal axes of the strain tensors $e$ and $\hat{e}$ are oriented along the same material fibers considered in the initial (for $e$) and current (for $\hat{e}$) states. Values of the principal components of the tensors $e_i$ and $\hat{e}_i$ are related by $1 + 2e_i = (1 - 2\hat{e}_i)^{-1}$ and change within $-0.5 < e_i < \infty$ and $-\infty < \hat{e}_i < 0.5$, respectively.

We introduce the tensors $\varepsilon = \varepsilon_{ij} \boldsymbol{l}^i \boldsymbol{l}^j$, $\hat{\varepsilon} = \hat{\varepsilon}_{ij} \hat{\boldsymbol{l}}^i \hat{\boldsymbol{l}}^j$, $\alpha = \alpha_{ij} \boldsymbol{l}^i \boldsymbol{l}^j$, and $\hat{\alpha} = \hat{\alpha}_{ij} \hat{\boldsymbol{l}}^i \hat{\boldsymbol{l}}^j$, which are coaxial with $e$ and $\hat{e}$. The covariant components of the tensors $\varepsilon$ and $\hat{\varepsilon}$ take values of the covariant components of the metric tensors $\varepsilon_{ij} = g_{ij} + 2e_{ij} = \hat{y}^m_{,x^i} \hat{y}^m_{,x^j}$ and $\hat{\varepsilon}_{ij} = \hat{g}_{ij} - 2\hat{e}_{ij} = y^m_{,\hat{x}^i} y^m_{,\hat{x}^j}$ of the accompanying coordinate systems. The contravariant components of the tensors $\alpha$ and $\hat{\alpha}$ are equal to the contravariant components of the metric tensors of the accompanying coordinate systems and to the components of the matrices $(G + 2E)^{-1}$ and $(\hat{G} - 2\hat{E})^{-1}$, i.e.,

$$\alpha^{ij} = [(G + 2E)^{-1}]^{ij} = x^i_{,\hat{y}^m} x^j_{,\hat{y}^m}, \quad \hat{\alpha}^{ij} = [(\hat{G} - 2\hat{E})^{-1}]^{ij} = \hat{x}^i_{,y^m} \hat{x}^j_{,y^m}. \tag{2.1}$$

The principal components of the tensors $\varepsilon_i$, $\hat{\varepsilon}_i$, $\alpha_i$, and $\hat{\alpha}_i$ are related by $\varepsilon_i = 1 + 2e_i = \hat{\alpha}_i$ and $\hat{\varepsilon}_i = 1 - 2\hat{e}_i = \alpha_i = \varepsilon_i^{-1}$. For every elementary material fiber, the ratio of the squared lengths in the initial and current states is $|d\hat{\boldsymbol{R}}|^2/|d\boldsymbol{R}|^2 = \varepsilon_{ij}\, dx^i\, dx^j/(g_{mn}\, dx^m\, dx^n)$ ($d\boldsymbol{R} = \boldsymbol{R}_{,x^i}\, dx^i$ and $d\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}_{,x^i}\, dx^i$). For the fibers oriented along the principal axes of the tensors $e$ and $\hat{e}$, this ratio takes extreme values equal to $\varepsilon_i$. Thus, the quantities $\varepsilon_i$ can be considered the squared principal extension ratios. For zero strain of the fiber, $\varepsilon_i = 1$; for unlimited contraction of the fiber, $\varepsilon_i \to 0$; for unlimited elongation, $\varepsilon_i \to \infty$.

**3. Stresses.** For an isotropic hyperelastic body, the stresses are determined from the strains at the current moment at the specified point using the following equations [1]:

$$\hat{\sigma}^{ij} = \hat{\mu}(\hat{\alpha}^{in} \hat{\alpha}^j_n - \hat{\chi}\hat{\alpha}^{ij}) + p\hat{g}^{ij}. \tag{3.1}$$

Here $\hat{\sigma}^{ij}$ are the contravariant components of the Cauchy stress tensor $\hat{\sigma} = \hat{\sigma}^{ij} \hat{\boldsymbol{l}}_i \hat{\boldsymbol{l}}_j$, $\hat{\mu} = \beta I_1^{-2} J^{-1}$, $\hat{\chi} = 2I_1(\Upsilon + 1/3)$, $\beta = \Psi_{,\Upsilon}$, and $p = \hat{\sigma}^n_n/3 = \Psi_{,J}$ is the hydrostatic pressure, where $\Psi$ is the strain energy density defined for materials as a function $\Psi = \Psi(\Upsilon, J)$. The strain-tensor invariants are expressed by $\Upsilon = I_2\, I_1^{-2}$, $I_1 = 3/2 + e^n_n = \hat{\alpha}^n_n/2$, $I_2 = e^{ij\prime} e'_{ij} = \hat{\alpha}^{ij\prime} \hat{\alpha}'_{ij}/4$ ($e^{ij\prime} = e^{ij} - e^n_n g^{ij}/3$ and $\hat{\alpha}^{ij\prime} = \hat{\alpha}^{ij} - \hat{\alpha}^n_n \hat{g}^{ij}/3$) and by Eqs. (1.2) for the Jacobian $J$. Equations (3.1) formulated for the second symmetric Piola–Kirchhoff stress tensor $\sigma = \sigma^{ij} \boldsymbol{l}_i \boldsymbol{l}_j$, where

$$\sigma^{ij} = J\hat{\sigma}^{mn} x^i_{,\hat{x}^m} x^j_{,\hat{x}^n}, \qquad \hat{\sigma}^{mn} = J^{-1} \sigma^{ij} \hat{x}^m_{,x^i} \hat{x}^n_{,x^j}, \tag{3.2}$$

become

$$\sigma^{ij} = \Psi_{,e_{ij}} = 2\mu(e^{ij\prime} - \chi g^{ij}) + \gamma\alpha^{ij}. \tag{3.3}$$

Here $\mu = \beta I_1^{-2}$, $\gamma = pJ$, and $\chi = I_1 \Upsilon$. For small strains, Eqs. (3.1) and (3.3) linearized with respect to strains, become relations of Hooke's law $\sigma^{ij} = 2\mu_0 e^{ij\prime} + K e^n_n\, g^{ij}$ with two material constants, namely, the shear modulus $\mu_0$ and the bulk modulus $K$ obtained at the limits $\mu \to \mu_0$, $p_{,J} \to K$ with strains tending to zero.

In (3.1), we convert to a Cartesian coordinate system with the coordinate axes directed along the principal axes of the tensors $\hat{e}$ and $\hat{\sigma}$. For the principal stress components $\hat{\sigma}_i$, we obtain the relations

$$\hat{\sigma}_i = \hat{\mu}\varepsilon_i(\varepsilon_i - \hat{\chi}) + p, \tag{3.4}$$

in which, determining $\hat{\mu}$ and $\hat{\chi}$ and the arguments of the function $\Psi$, we can use the representations

$$I_1 = \frac{1}{2}(\varepsilon_m + \varepsilon_n + \varepsilon_l), \quad J = (\varepsilon_m \varepsilon_n \varepsilon_l)^{1/2}, \quad \Upsilon = 2\Big(\frac{1}{3} - \frac{\varepsilon_m\varepsilon_n + \varepsilon_n\varepsilon_l + \varepsilon_l\varepsilon_m}{(\varepsilon_m + \varepsilon_n + \varepsilon_l)^2}\Big)$$

($m, n, l$ is even rearrangement of the subscripts and superscripts 1, 2, and 3). As $\varepsilon_i \geqslant 0$, the value of $\Upsilon$ satisfies the condition $0 \leqslant \Upsilon \leqslant 2/3$ and can be regarded as a measure of inequality of the principal extension ratios. In the space of Cartesian coordinates $\varepsilon_i \geqslant 0$, the value of $\Upsilon$ is the one third of the squared slope angle between the ray issuing from the origin to the given point $\varepsilon_i$ and the ray along which $\varepsilon_m = \varepsilon_n = \varepsilon_l$. At $\varepsilon_m = \varepsilon_n = \varepsilon_l$, $\Upsilon = 0$ and at $\varepsilon_m = \varepsilon_n = 0$, $\Upsilon = 2/3$.

For isotropic hyperelastic materials, as an additional condition we can assume that the maximum stress $\hat{\sigma}_i$ must act in the direction of the principal axis with the maximum extension $\varepsilon_i$. In this case, in accordance with (3.4), $\beta > 0$.

**4. Variation in the Areas of Elementary Material Sites and Normals to Them.** In an undeformed body, we consider a parallelogram site with adjacent sides $d\boldsymbol{R}^{(1)} = \boldsymbol{R}_{,x^i}\, dx^{i(1)}$ and $d\boldsymbol{R}^{(2)} = \boldsymbol{R}_{,x^i}\, dx^{i(2)}$ and area $dS$. The unit normal to this site $\boldsymbol{N} = N_i \boldsymbol{l}^i$ is given by the equality

$$\boldsymbol{N}\, dS = d\boldsymbol{R}^{(1)} \times d\boldsymbol{R}^{(2)} = (\det G)^{1/2} c_{ijk} \boldsymbol{l}^k\, dx^{i(1)}\, dx^{j(2)},$$

where $c_{ijk}$ is an antisymmetric object [2] that takes values of 1 and $-1$ if $i$, $j$, and $k$ are even and uneven rearrangements of the subscripts and superscripts 1, 2, and 3, respectively; otherwise it is equal to zero. At the current moment, the initial site becomes a parallelogram with sides $d\hat{\boldsymbol{R}}^{(1)} = \hat{\boldsymbol{R}}_{,x^i}\, dx^{i(1)}$ and $d\hat{\boldsymbol{R}}^{(2)} = \hat{\boldsymbol{R}}_{,x^i}\, dx^{i(2)}$, area $d\hat{S}$, and unit normal $\hat{\boldsymbol{N}} = \hat{N}_i \hat{\boldsymbol{l}}^i$, for which

$$\hat{\boldsymbol{N}} d\hat{S} = d\hat{\boldsymbol{R}}^{(1)} \times d\hat{\boldsymbol{R}}^{(2)} = \det \hat{\Gamma}\, (\det \hat{G})^{1/2} c_{ijk} x^k_{,\hat{x}^m} \hat{\boldsymbol{l}}^m\, dx^{i(1)}\, dx^{j(2)}. \tag{4.1}$$

Determining the matrix determinant $\hat{\Gamma} = \|\hat{x}^i_{,x^j}\|$ from (1.1) and (1.2): $\hat{\Gamma}^{\rm t} \hat{G} \hat{\Gamma} = G + 2E$, $\det \hat{\Gamma} = J\, (\det \hat{G})^{-1/2} (\det G)^{1/2}$, we obtain [3]

$$\hat{N}_m\, d\hat{S} = J N_i x^i_{,\hat{x}^m}\, dS. \tag{4.2}$$

At the moment $\tau$, we now consider another site with adjacent sides $d\hat{\boldsymbol{R}}^{(1)\prime} = \hat{\boldsymbol{R}}_{,\hat{x}^i}\, d\hat{x}^{i(1)}$ and $d\hat{\boldsymbol{R}}^{(2)\prime} = \hat{\boldsymbol{R}}_{,\hat{x}^i}\, d\hat{x}^{i(2)}$ and an orthogonal vector whose absolute value is equal to the area of the site:

$$\hat{\boldsymbol{N}}\, d\hat{S} = d\hat{\boldsymbol{R}}^{(1)\prime} \times d\hat{\boldsymbol{R}}^{(2)\prime} = (\det \hat{G})^{1/2} c_{ijk} \hat{\boldsymbol{l}}^k\, d\hat{x}^{i(1)}\, d\hat{x}^{j(2)}.$$

Here $d\hat{S}$ and $\hat{\boldsymbol{N}}$ can differ from those given by equality (4.1). At the moment $\tau + \Delta\tau$, the sides of the site take positions $d\hat{\boldsymbol{R}}^{(1)\prime}_{\tau+\Delta\tau} = (\hat{\boldsymbol{R}} + \boldsymbol{v}\Delta\tau)_{,\hat{x}^i}\, d\hat{x}^{i(1)}$ and $d\hat{\boldsymbol{R}}^{(2)\prime}_{\tau+\Delta\tau} = (\hat{\boldsymbol{R}} + \boldsymbol{v}\Delta\tau)_{,\hat{x}^i}\, d\hat{x}^{i(2)}$. For the site vector, we obtain $\hat{\boldsymbol{N}}_{\tau+\Delta\tau}\, d\hat{S}_{\tau+\Delta\tau} = d\hat{\boldsymbol{R}}^{(1)\prime}_{\tau+\Delta\tau} \times d\hat{\boldsymbol{R}}^{(2)\prime}_{\tau+\Delta\tau} = [(1 + \hat{\eta}^m_m \Delta\tau)\hat{\boldsymbol{N}} - (\boldsymbol{v}_{,\hat{x}^i} \cdot \hat{\boldsymbol{N}})\hat{\boldsymbol{l}}^i \Delta\tau]\, d\hat{S}$. From these expressions, letting $\Delta\tau \to 0$, we find the rate of variation of the areas of the sites and normals to them:

$$(d\hat{S})^{\dot{}} = (\hat{\eta}^m_m - \hat{\eta}_{ij} \hat{N}^i \hat{N}^j)\, d\hat{S}, \qquad (\hat{\boldsymbol{N}})^{\dot{}} = (\hat{\eta}_{ij} \hat{N}^i \hat{N}^j)\hat{\boldsymbol{N}} - (\boldsymbol{v}_{,\hat{x}^i} \cdot \hat{\boldsymbol{N}})\hat{\boldsymbol{l}}^i. \tag{4.3}$$

The rates of rotation of the normals $(\hat{\boldsymbol{N}})^{\dot{}}$ depend on the direction and rate of rigid-body rotation of the neighborhood of the material point.

**5. Determination of Forces.** Force vectors acting at the sites represented at the initial and current moments by the vectors $\boldsymbol{N}\, dS$ and $\hat{\boldsymbol{N}}\, d\hat{S}$, respectively, are given by

$$\hat{\boldsymbol{q}}\, d\hat{S} = \hat{\sigma}^{ij} \hat{N}_j \hat{\boldsymbol{R}}_{,\hat{x}^i}\, d\hat{S} = \sigma^{ij} N_j \hat{\boldsymbol{R}}_{,x^i}\, dS, \tag{5.1}$$

where $\hat{\boldsymbol{q}}$ is the force density per unit area of the site. At Cartesian sites with normals $\boldsymbol{k}_m = \hat{y}^m_{,\hat{x}^i} \hat{\boldsymbol{l}}^i = \hat{x}^i_{,\hat{y}^m} \hat{\boldsymbol{l}}_i$, the force densities take values $\hat{\boldsymbol{q}}^{(m)} = \hat{\sigma}^{ij} \hat{y}^m_{,\hat{x}^j} \hat{\boldsymbol{R}}_{,\hat{x}^i}$. The projections of the reduced vectors onto the normals and tangents to the sites $\hat{\sigma}^{mn(d)} = \hat{\boldsymbol{q}}^{(m)} \cdot \boldsymbol{k}_n = \hat{\sigma}^{ij} \hat{y}^m_{,\hat{x}^j} \hat{y}^n_{,\hat{x}^i} = J^{-1} \sigma^{ij} \hat{y}^m_{,x^i} \hat{y}^n_{,x^j}$ are the physical, true, stresses acting on the Cartesian coordinate sites, i.e., the Cartesian stress-tensor components $\hat{\sigma} = \hat{\sigma}^{mn(d)} \boldsymbol{k}_m \boldsymbol{k}_n$.

**6. Stress Rates.** Let us determine the stress rates taking into account the rigid-body rotation of the neighborhood of the material point and the rotation of the sites at which the examined stresses occur. We assume that at the moment $\tau$, the Cartesian coordinate sites are rotating together with the neighborhood of the point with angular velocity vector $\boldsymbol{\omega} = \omega^i \boldsymbol{k}_i$. At the moment $\tau + \Delta\tau$, the sites take positions with the normals

$$\hat{\boldsymbol{N}}^{(m)}_{\tau+\Delta\tau} = \boldsymbol{k}_m + \boldsymbol{\omega} \times \boldsymbol{k}_m\, \Delta\tau = \boldsymbol{k}_m + (\boldsymbol{k}_n \omega^l - \boldsymbol{k}_l \omega^n)\, \Delta\tau. \tag{6.1}$$

Here and below the subscript and superscript $m$ correspond to the values obtained for the site with the normal $\boldsymbol{k}_m$, terms of order $(\Delta\tau)^2$ are neglected, and $m, n, l$ is even rearrangement of the subscripts and superscripts 1, 2, and 3. We consider elementary material sites with normals $\hat{\boldsymbol{N}}^{(m)}_{\tau+\Delta\tau}$ and areas $d\hat{S}^{(m)}_{\tau+\Delta\tau}$ at the moment $\tau + \Delta\tau$.

Substituting $\hat{\boldsymbol{N}} = \boldsymbol{k}_m$ into (4.3), we determine $(d\hat{S})^{\boldsymbol{\cdot}(m)} = (\hat{\eta}^i_i - \hat{\eta}^{(d)}_m)\, d\hat{S}^{(m)}$, $(\hat{\boldsymbol{N}})^{\boldsymbol{\cdot}(m)} = \hat{\eta}^{(d)}_m \boldsymbol{k}_m - (\boldsymbol{v}_{,\hat{x}^i} \cdot \boldsymbol{k}_m)\hat{\boldsymbol{l}}^i$, and $\hat{\eta}^{(d)}_m = \hat{\eta}_{ij}\hat{x}^i_{,\hat{y}^m}\hat{x}^j_{,\hat{y}^m}$ (summation over $m$ is not performed). From the equalities $d\hat{S}^{(m)}_{\tau+\Delta\tau} = d\hat{S}^{(m)} + (d\hat{S})^{\boldsymbol{\cdot}(m)}\Delta\tau$ and $\hat{\boldsymbol{N}}^{(m)}_{\tau+\Delta\tau} = \hat{\boldsymbol{N}}^{(m)} + (\hat{\boldsymbol{N}})^{\boldsymbol{\cdot}(m)}\Delta\tau$, we find approximate values of the areas and normals to the sites at the moment $\tau$:

$$
\begin{gathered}
d\hat{S}^{(m)} = [1 - (\hat{\eta}^i_i - \hat{\eta}^{(d)}_m)\,\Delta\tau]\, d\hat{S}^{(m)}_{\tau+\Delta\tau}, \\
\hat{\boldsymbol{N}}^{(m)} = (1 - \hat{\eta}^{(d)}_m\,\Delta\tau)\boldsymbol{k}_m + [\boldsymbol{k}_n\omega^l - \boldsymbol{k}_l\omega^n + (\boldsymbol{v}_{,\hat{x}^i} \cdot \boldsymbol{k}_m)\hat{\boldsymbol{l}}^i]\,\Delta\tau.
\end{gathered} \tag{6.2}
$$

According to (4.2) and (6.2), at the moment $\tau = 0$, the sites have areas $dS^{(m)}$ and unit normals $\boldsymbol{N}^{(m)} = N^{(m)}_i \boldsymbol{l}^i$, where

$$
N^{(m)}_i = \frac{d\hat{S}^{(m)}}{J\, dS^{(m)}}\,\{(1 - \hat{\eta}^{(d)}_m\,\Delta\tau)\hat{y}^m_{,x^i} + [\omega^l\hat{y}^n_{,x^i} - \omega^n\hat{y}^l_{,x^i} + (\boldsymbol{v}_{,x^i} \cdot \boldsymbol{k}_m)]\,\Delta\tau\}. \tag{6.3}
$$

For the sites with the normals (6.1) determined at the moment $\tau + \Delta\tau$, the force densities determined from (5.1) are

$$
\hat{\boldsymbol{q}}^{(m)}_{\tau+\Delta\tau} = (\sigma^{ij} + \dot{\sigma}^{ij}\,\Delta\tau)(\hat{\boldsymbol{R}} + \boldsymbol{v}\,\Delta\tau)_{,x^i} N^{(m)}_j\, dS^{(m)}\,(d\hat{S}^{(m)}_{\tau+\Delta\tau})^{-1}.
$$

With allowance for (6.2) and (6.3), they are expressed as

$$
\begin{aligned}
\hat{\boldsymbol{q}}^{(m)}_{\tau+\Delta\tau} = \{(1 - \hat{\eta}^k_k\Delta\tau)\hat{y}^m_{,x^j} + [\omega^l\hat{y}^n_{,x^j} - \omega^n\hat{y}^l_{,x^j} + (\boldsymbol{v}_{,x^j} \cdot \boldsymbol{k}_m)]\,\Delta\tau\}J^{-1}\sigma^{ij}\hat{\boldsymbol{R}}_{,x^i} \\
+ (\sigma^{ij}\boldsymbol{v}_{,x^i} + \dot{\sigma}^{ij}\hat{\boldsymbol{R}}_{,x^i})J^{-1}\hat{y}^m_{,x^j}\,\Delta\tau.
\end{aligned}
$$

Using the Cartesian components of the tensors $\hat{\sigma}$ and

$$
\hat{s} = J^{-1}\dot{\sigma}^{ij}\hat{\boldsymbol{R}}_{,x^i}\hat{\boldsymbol{R}}_{,x^j} = \hat{s}^{ij}\hat{\boldsymbol{l}}_i\hat{\boldsymbol{l}}_j = \hat{s}^{ij(d)}\boldsymbol{k}_i\boldsymbol{k}_j, \tag{6.4}
$$

we obtain the following expressions for the force densities considered:

$$
\hat{\boldsymbol{q}}^{(m)}_{\tau+\Delta\tau} = (1 - \hat{\eta}^k_k\,\Delta\tau)\hat{\sigma}^{mi(d)}\boldsymbol{k}_i + [\omega^l\hat{\sigma}^{ni(d)} - \omega^n\hat{\sigma}^{li(d)} + (\boldsymbol{v}_{,\hat{y}^j} \cdot \boldsymbol{k}_m)\hat{\sigma}^{ji(d)}]\boldsymbol{k}_i\,\Delta\tau + (\hat{\sigma}^{mi(d)}\boldsymbol{v}_{,\hat{y}^i} + \hat{s}^{mi(d)}\boldsymbol{k}_i)\,\Delta\tau.
$$

The projections of these vectors onto the normal and tangents to the sites (6.1)

$$
\hat{\sigma}^{ij(d)}_{\tau+\Delta\tau} = \hat{\boldsymbol{q}}^{(i)}_{\tau+\Delta\tau} \cdot \hat{\boldsymbol{N}}^{(j)}_{\tau+\Delta\tau} = \hat{\sigma}^{ij(d)} + (\hat{s}^{ij(d)} + \hat{\sigma}^{ik(d)}\hat{\eta}^{(d)}_{kj} + \hat{\sigma}^{kj(d)}\hat{\eta}^{(d)}_{ik} - \hat{\sigma}^{ij(d)}\hat{\eta}^k_k)\,\Delta\tau
$$

are approximate values of the physical true stresses that act at the material point considered in a small time $\Delta\tau$ instead of the stresses $\hat{\sigma}^{ij(d)}$ acting at it at the moment $\tau$. Letting $\Delta\tau$ to zero, we determine the stress rates

$$
\hat{\Sigma}^{ij(d)} = \lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\,(\hat{\sigma}^{ij(d)}_{\tau+\Delta\tau} - \hat{\sigma}^{ij(d)}) = \hat{s}^{ij(d)} + \hat{\sigma}^{ik(d)}\hat{\eta}^{(d)}_{kj} + \hat{\sigma}^{kj(d)}\hat{\eta}^{(d)}_{ik} - \hat{\sigma}^{ij(d)}\hat{\eta}^k_k,
$$

which are the Cartesian components of the symmetric tensor $\hat{\Sigma} = \hat{\Sigma}^{ij(d)}\boldsymbol{k}_i\boldsymbol{k}_j = \hat{\Sigma}^{ij}\hat{\boldsymbol{l}}_i\hat{\boldsymbol{l}}_j$. Resolving into the basis vectors of the curvilinear coordinate system taken at the current position of the material point, we have

$$
\hat{\Sigma}^{ij} = \hat{s}^{ij} + \hat{\sigma}^{ik}\hat{\eta}^j_k + \hat{\sigma}^{kj}\hat{\eta}^i_k - \hat{\sigma}^{ij}\hat{\eta}^k_k. \tag{6.5}
$$

Writing the Cauchy stress tensor as $\hat{\sigma} = J^{-1}\sigma^{ij}(\boldsymbol{l}_i + \boldsymbol{u}_{,x^i})(\boldsymbol{l}_j + \boldsymbol{u}_{,x^j})$ and differentiating it with respect to time $\tau$, we obtain the expression for $\hat{\Sigma}$ as the Jaumann stress-rate tensor [3–5], $\hat{\Sigma} = (\hat{\sigma})^{\boldsymbol{\cdot}} + \hat{\sigma} \cdot \hat{\Omega} - \hat{\Omega} \cdot \hat{\sigma}$, where $\hat{\Omega} = \hat{\Omega}_{ij}\hat{\boldsymbol{l}}^i\hat{\boldsymbol{l}}^j$, $\hat{\Omega}_{ij} = (\hat{v}_{i;j} - \hat{v}_{j;i})/2$, and $\hat{\sigma} \cdot \hat{\Omega} = \hat{\sigma}^{ik}\hat{\Omega}_{kj}\hat{\boldsymbol{l}}_i\hat{\boldsymbol{l}}^j$. The tensor $\hat{\Sigma}$ is determined irrespective of the properties of the material.

**7. Equations Relating the Stress Rates to the Strain Rates in an Isotropic Hyperelastic Body.** Differentiating Eq. (3.3) with respect to $\tau$, we obtain

$$\dot{\sigma}^{ij} = [(\dot{J}p_{,J} + \dot{\Upsilon}p_{,\Upsilon})J + p\dot{J}]\alpha^{ij} + [(\dot{J}\beta_{,J} + \dot{\Upsilon}\beta_{,\Upsilon})\beta^{-1} - 2I_1^{-1}\dot{I}_1]\sigma^{(1)ij}$$
$$+2\mu(\eta^{ij\prime} - \dot{\chi}g^{ij}) - 2pJ\alpha^{im}\alpha^{jn}\eta_{mn}, \qquad (7.1)$$

where $p_{,\Upsilon} = \beta_{,J}$, $\sigma^{(1)ij} = \sigma^{ij} - pJ\alpha^{ij}$, and $\eta^{ij\prime} = \eta^{ij} - g^{ij}\eta_m^m/3$. Using (6.4) and taking into account (1.3), (2.1), (3.1), (3.2), and (7.1), we express $\hat{s}^{ij}$ in terms of $\dot{\sigma}^{ij}$ and substitute the result into (6.5). We obtain the stress rate equations

$$\hat{\Sigma}^{ij} = (\dot{J}p_{,J} + \dot{\Upsilon}p_{,\Upsilon})\hat{g}^{ij} + [(\dot{J}\beta_{,J} + \dot{\Upsilon}\beta_{,\Upsilon})\beta^{-1} - 2I_1^{-1}\dot{I}_1]\hat{\sigma}^{ij\prime}$$
$$+ 2\hat{\mu}\hat{\alpha}^{im}\hat{\alpha}^{jn}\hat{\eta}_{mn} - \hat{\mu}\hat{\alpha}^{ij}(\hat{\chi})^{\cdot} + \hat{\sigma}^{in\prime}\hat{\eta}_n^j + \hat{\sigma}^{nj\prime}\hat{\eta}_n^i - \hat{\sigma}^{ij\prime}\hat{\eta}_m^m, \qquad (7.2)$$

which are linear (after substitutions $\dot{I}_1 = \hat{\alpha}^{ij}\hat{\eta}_{ij}$, $\dot{J} = J\hat{\eta}_i^i$, $\dot{\Upsilon} = \beta^{-1}J\hat{\sigma}^{ij\prime}\hat{\eta}_{ij}$, and $\hat{\sigma}^{ij\prime} = \hat{\sigma}^{ij} - p\hat{g}^{ij}$) with respect to the strain rates $\hat{\eta}_{ij}$ with a nonsymmetric matrix of the coefficients.

In a Cartesian coordinate system with the axes directed along the principal axes of the tensors $\hat{\sigma}$ and $\hat{e}$, from (7.2), we obtain the expressions for the normal stress rates (summation over $i$ is not performed)

$$\hat{\Sigma}^{ii} = \dot{J}p_{,J} + \dot{\Upsilon}p_{,\Upsilon} + [(\dot{J}\beta_{,J} + \dot{\Upsilon}\beta_{,\Upsilon})\beta^{-1} - 2I_1^{-1}\dot{I}_1 + 2\hat{\eta}_{ii} - \hat{\eta}_k^k]\hat{\sigma}_i' + \hat{\mu}\varepsilon_i[2\varepsilon_i\hat{\eta}_{ii} - (\hat{\chi})^{\cdot}] \qquad (\hat{\sigma}_i' = \hat{\sigma}_i - p) \qquad (7.3)$$

and shear stress rates ($m, n, l$ is even rearrangement of the subscripts and superscripts 1, 2, and 3)

$$\hat{\Sigma}^{mn} = B_l\,\hat{\eta}_{mn}, \qquad B_l = 2\hat{\mu}\varepsilon_m\varepsilon_n - \hat{\sigma}_l' = \hat{\mu}(\varepsilon_m + \varepsilon_n)[2\varepsilon_m\varepsilon_n + \varepsilon_l(\varepsilon_m + \varepsilon_n) - \varepsilon_l^2]/(2I_1). \qquad (7.4)$$

For positive values of $B_l > 0$, occurring, in particular, under small strains if $B_l \approx 2\mu_0$, the values of $\hat{\Sigma}^{mn}$ and $\hat{\eta}_{mn}$ have the same signs. If $B_l = 0$ and $\hat{\Sigma}^{mn} = 0$, the rates $\hat{\eta}_{mn}$ are not determined uniquely from (7.4). If $B_l < 0$, the shear stress rate and the corresponding shear strain rate should be in opposition. Therefore, satisfaction of the inequality $B_l > 0$ can be considered as a condition of stable deformation of the material. At each time, just one of the coefficients $B_l$ can be equal to zero because for $\hat{\sigma}_l' = 2\hat{\mu}\varepsilon_m\varepsilon_n$, it must be $\hat{\sigma}_m' = \hat{\sigma}_n' = -\hat{\mu}\varepsilon_m\varepsilon_n < 0$. Reducing $B_l$ by nonnegative factors, we obtain the conditions of stable deformation

$$\varepsilon_l < (\varepsilon_m + \varepsilon_n)/2 + \sqrt{(\varepsilon_m + \varepsilon_n)^2/4 + 2\varepsilon_m\varepsilon_n},$$

which impose restrictions on the values of the squared principal extension ratios $\varepsilon_m$, $\varepsilon_n$, and $\varepsilon_l$ and do not depend on the form of the constitutive function $\Psi$ and values of any material constants.

In matrix form, the Eq. (7.3) is written as

$$Z = DX, \qquad (7.5)$$

where $X^{\mathrm{t}} = (\hat{\eta}_{11}, \hat{\eta}_{22}, \hat{\eta}_{33})$, $Z^{\mathrm{t}} = (\hat{\Sigma}^{11}, \hat{\Sigma}^{22}, \hat{\Sigma}^{33})$, and $D$ is an asymmetric matrix of coefficients that depends only on the values of $\varepsilon_i$ at the current time. In the undeformed state of the material, the matrix $D$ is symmetric and positively determined, and Eq. (7.5) is uniquely solvable with respect to the strain rates. Under loading at $\det D = 0$, the solution of Eq. (7.5) can be nonunique and should be chosen with allowance for the existing restrictions on the permissible stress and strain rates. At subsequent times, $D$ can have negative eigenvalues.

For the solutions $X$ and $Z$ of Eq. (7.5) with components of opposite signs, a decreasing stress–strain relation (declining curve) occurs for every principal axis: elongation of the fiber directed along the axis is accompanied by a decrease in the stress acting in the fiber (summation over $i$ is not performed): $\hat{\eta}_{ii} \geqslant 0$ and $\hat{\Sigma}^{ii} \leqslant 0$; contraction of the fiber is accompanied by an increase in the stress: $\hat{\eta}_{ii} \leqslant 0$ and $\hat{\Sigma}^{ii} \geqslant 0$. Such deformation of the material is considered unstable. Below, we study the deformation of an isotropic elastic material under certain simple loads with the function $\Psi$ derived by generalization of the constitutive function of Hooke's law. In the solutions given below, the eigenvalues of the matrix $D$ are only real.

**8. Deformation of an Isotropic Hyperelastic Material under Simple Loading.** In Hooke's law, the stresses $\sigma^{ij} = 2\mu_0 e^{ij\prime} + Ke_n^n g^{ij}$, where $\mu_0 = E_0/[2(1+\nu)]$ and $K = E_0/[3(1-2\nu)]$ ($E_0$ is Young's modulus and $\nu$ is Poisson's ratio) are partial derivatives $\sigma^{ij} = \Psi_{0,e_{ij}}$ of the function $\Psi_0 = \Psi_0(I_2, J_1) = \mu_0 I_2 + 0.5K(J_1 - 1)^2$, which depends on two arguments $I_2 = e^{ij\prime}e_{ij}'$ and $J_1 = 1 + e_n^n$. Under small strains, $J_1$ and $I_2$ can be treated as approximate representations of the arguments $J$ and $\Upsilon = I_2 I_1^{-2}$ of the constitutive function $\Psi$ for an isotropic hyperelastic material if we take into account that $J \approx J_1$ and omit the factor with $I_1 \approx 3/2$ in the expression for $\Upsilon$.

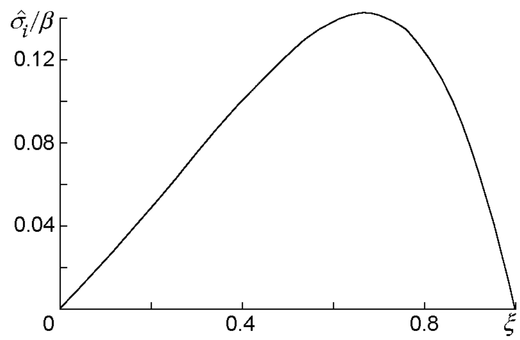


Fig. 1

Let us determine the function $\Psi$ which continuously becomes $\Psi_0$ if the strain tends to zero. We specify the pressure dependence in the form $p = 0.5K(1 - J^{-2})$. With decrease in the volume of the material, the pressure increases in absolute value without bound; as $J \to \infty$, the pressure tends to the maximum permissible value and as $J \to 1$, it approaches the value determined by Hooke's law. A given $p$ corresponds to the function $\Psi_2 = \displaystyle\int p\,dJ = 0.5KJ^{-1}(J-1)^2$ $(p = \Psi_{2,J})$. To $\Psi_2$, we add the term $\beta\Upsilon$ with the coefficient $\beta = 9\mu_0/4$, which approximates the first term in $\Psi_0$. As a result, for an isotropic hyperelastic body, we obtain the constitutive function $\Psi = \beta\Upsilon + 0.5KJ^{-1}(J-1)^2$ with the two material constants $\mu_0$ and $K$ as in Hooke's law. In the thus determined function $\Psi$, zeroes of the eigenvalues of the matrix $D$, for which, according to Eq. (7.5), there are ramification of solutions and transition to decreasing stress–strain relations, occur at moments independent of the values of $\mu_0$ and $K$.

Pure shear loading occurs for $\hat{\sigma}_i = -\hat{\sigma}_j$, $\hat{\sigma}_k = p = 0$, and $J = 1$. From Eqs. (3.4), we determine $\varepsilon_k = (\varepsilon_i^2 + \varepsilon_j^2)/(\varepsilon_i + \varepsilon_j)$ and $\hat{\sigma}_i = \beta\varepsilon_i\varepsilon_j(\varepsilon_i^2 - \varepsilon_j^2)/(\varepsilon_i^2 + \varepsilon_i\varepsilon_j + \varepsilon_j^2)^2$. Taking into account that $\hat{\sigma}_i > 0$ and it is a homogeneous zero-order function of $\varepsilon_i$ and $\varepsilon_j$ and considering the monotonic increase in $\varepsilon_i$ and decrease in $\varepsilon_j$, we introduce the parameter $0 \leqslant \xi = 1 - \varepsilon_j/\varepsilon_i \leqslant 1$ and obtain the dependences $\varepsilon_i = \{(2-\xi)/[(1-\xi)(2-2\xi+\xi^2)]\}^{1/3}$, $\varepsilon_j = [(2-\xi)(1-\xi)^2/(2-2\xi+\xi^2)]^{1/3}$, and $\varepsilon_k = \{(2-2\xi+\xi^2)^2/[(1-\xi)(2-\xi)^2]\}^{1/3}$. We have $\varepsilon_i \to \infty$ and $\varepsilon_j \to 0$, and the value of $\varepsilon_k$ approaches $\varepsilon_i$ as $\xi \to 1$. In the current state, an elementary material particle which initially is a cube with edges directed along the principal axes of the tensor $\sigma$ and $e$ takes the form of a right parallelepiped with one edge contracting without bound and the other two elongating without bound. The particle volume remains constant.

The stresses $\hat{\sigma}_i$ increase (see Fig. 1) up to the time when $\xi = \xi_* = 0.671457$, $\varepsilon_i = 1.53465$, $\varepsilon_j = 0.505842$, and $\varepsilon_k = 1.28399$, at which the maximum (of the order of $\mu_0$) value of $\hat{\sigma}_i = 0.142031\beta$ is reached. In the region $\xi_* < \xi < 1$, an eigenvalue of the matrix $D$, vanishing at $\xi = \xi_*$, becomes negative. As a result, the curve in Fig. 1 becomes declining; the stresses decrease monotonically [and so do the forces acting on the cube edges, which are required to keep the cube in the strained state and are proportional to the quantities $\hat{\sigma}_j(\varepsilon_k\varepsilon_i)^{1/2}$ and $\hat{\sigma}_i(\varepsilon_j\varepsilon_k)^{1/2}$], and the deformation of the material is considered unstable. Throughout the entire deformation process, $B_l > 0$. Over a wide range of variation of the values of $\hat{\sigma}_i$ up to values of the order of the shear modulus, the dependence of $\hat{\sigma}_i$ on $\varepsilon_i$ (solid curve in Fig. 2) is nearly linear, as follows from Hooke's law $\hat{\sigma}_i = 2\mu_0 e_i$ (dashed curve in Fig. 2). For Hooke's law, the length of the transverse fiber remains constant: $\varepsilon_k = 1$.

In the case of axisymmetric loading $\hat{\sigma}_i = \hat{\sigma}_j$ and $\hat{\sigma}_k = 0$, according to (3.4), we have

$$\hat{\mu}(\varepsilon_i - \varepsilon_j)(\varepsilon_i + \varepsilon_j - \hat{\chi}) = 0, \qquad \hat{\mu}\varepsilon_k(\varepsilon_k - \hat{\chi}) + p = 0, \qquad \hat{\sigma}_i = 1.5p. \tag{8.1}$$

Of the two solutions of the first equation in (8.1), only one solution corresponding to the axisymmetric strain $\varepsilon_i = \varepsilon_j$ can occur at the beginning of loading. Introducing the parameter $-1 \leqslant \xi = (\varepsilon_i - \varepsilon_k)/(\varepsilon_i + \varepsilon_k) \leqslant 1$, we obtain the dependences $\gamma = 16\beta\xi(1-\xi^2)/(3+\xi)^3$, $J = \gamma/K + \sqrt{(\gamma/K)^2 + 1}$, $\varepsilon_i = J^{2/3}[(1+\xi)/(1-\xi)]^{1/3}$, $\varepsilon_k = J^{2/3}[(1-\xi)/(1+\xi)]^{2/3}$, and $p = \gamma/J$. Below, we give results of calculations for $\nu = 0.3$. In this axisymmetric solution (solid curve in Fig. 3), the stress reaches extreme values for tension $(0 \leqslant \xi \leqslant 1)$ if $\xi = \xi_*^+ = (2\sqrt{7} - 1)/9$, $\varepsilon_i = 1.55687$, $\varepsilon_k = 0.551518$, $J = 1.1562$, and $\hat{\sigma}_i = 0.157462E_0$ and for compression $(-1 \leqslant \xi \leqslant 0)$ if $\xi = \xi_*^- = -(2\sqrt{7} + 1)/9$, $\varepsilon_i = 0.410525$, $\varepsilon_k = 2.31772$, $J = 0.624986$, and $\hat{\sigma}_i = -0.975072E_0$. As $\xi \to \pm 1$, the volume of the material becomes initial: $J \to 1$ and $\hat{\sigma}_i \to 0$.

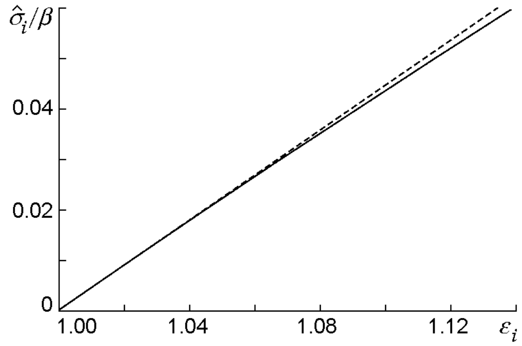


Fig. 2

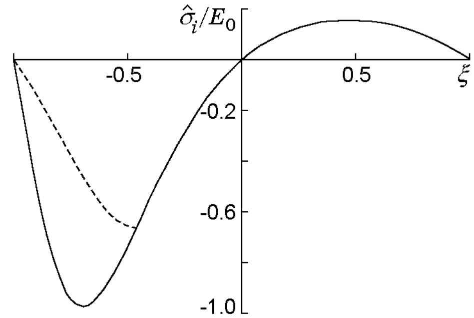


Fig. 3

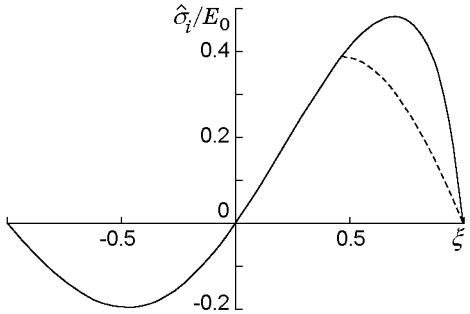


Fig. 4

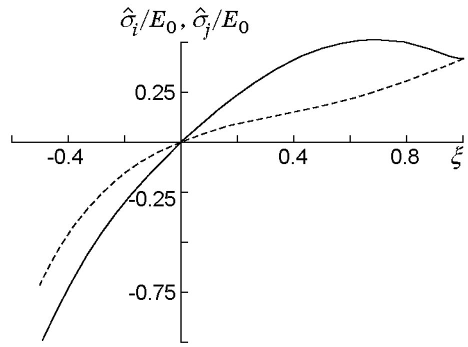


Fig. 5

During the compression process, where $\xi = \xi_{**} = 3 - 2\sqrt{3}$, $\varepsilon_i = 0.561577$, $\varepsilon_k = 1.53426$, $J = 0.695598$, and $\hat{\sigma}_i = -0.666704E_0$, one of the eigenvalues of the matrix $D$ vanishes and a nonaxisymmetric solution determined from (8.1) branches off from the axisymmetric solution: $\varepsilon_i + \varepsilon_j = \hat{\chi}$, $\varepsilon_k = (\varepsilon_i + \varepsilon_j)/2 + \sqrt{(\varepsilon_i + \varepsilon_j)^2/4 + 2\varepsilon_i\varepsilon_j}$, and $p = -2\hat{\mu}\varepsilon_i\varepsilon_j$. Introducing $\zeta_1 = (\varepsilon_i + \varepsilon_j)/2$ and $\zeta_2 = 2\zeta_1^{-2}\varepsilon_i\varepsilon_j$, we obtain the expressions $\gamma = -4\beta\zeta_2/(3 + \sqrt{1+\zeta_2})^2$, $J = \gamma/K + \sqrt{(\gamma/K)^2 + 1}$, $\zeta_1 = \{2J^2/[\zeta_2\Big(1 + \sqrt{1+\zeta_2}\Big)]\}^{1/3}$, $\varepsilon_i = \zeta_1(1 - \sqrt{1 - \zeta_2/2})$, $\varepsilon_j = \zeta_1(1 + \sqrt{1 - \zeta_2/2})$, $\varepsilon_k = \zeta_1(1 + \sqrt{1+\zeta_2})$, and $p = \gamma/J$, in which $0 \leqslant \zeta_2 \leqslant 2$. As $\zeta_2 \to 0$, we have $\varepsilon_i \to 0$ and $\varepsilon_j \to \infty$. The volume of the material increases monotonically, approaching the initial value, and the stress tends to zero ($J \to 1$ and $\hat{\sigma}_i \to 0$ as $\zeta_2 \to 0$) (dashed curve in Fig. 3). Under nonaxisymmetric deformation, the stress state ($\hat{\sigma}_i = \hat{\sigma}_j$) remains axisymmetric. For $-1 < \xi < \xi_{**}$ and $\xi_*^+ < \xi < 1$, the nonaxisymmetric deformation, as well as the axisymmetric deformation, is unstable and occurs with decreasing absolute values of the stresses $\hat{\sigma}_i$, which are smaller than those in the axisymmetric solution. The values of $\xi_*^+$, $\xi_*^-$, and $\xi_{**}$ are independent of the material's constants.

We consider an elementary material particle which initially has the shape of a cube with edges directed along the principal axes of the tensors $\sigma$ and $e$. Under axisymmetric tensile strain at the limit, it becomes a square plate with an unrestrictedly decreasing thickness and elongating sides. Under axisymmetric compressive strain, it becomes an unrestrictedly elongating prism with square cross section and surface area approaching zero. In the nonaxisymmetric solution branches off under compression, the prism cross section is no longer square: one side of the cross section lengthens unrestrictedly, and the other side shortens unrestrictedly. The prism becomes a square plate located on the axial plane. At the limit, the forces required to keep the cube in the strained state tend to zero as well as the stresses.

Similar results are obtained for the case of uniaxial loading by stress $\hat{\sigma}_i$ for $\hat{\sigma}_j = \hat{\sigma}_k = 0$. In the axisymmetric solution $\varepsilon_j = \varepsilon_k$, $-1 \leqslant \xi = (\varepsilon_i - \varepsilon_j)/(\varepsilon_i + \varepsilon_j) \leqslant 1$ (solid curve in Fig. 4) determined from the equations $\hat{\mu}(\varepsilon_j - \varepsilon_k)(\varepsilon_j + \varepsilon_k - \hat{\chi}) = 0$, $p = 0.5\hat{\mu}\varepsilon_i(\varepsilon_i - \hat{\chi})$, and $\hat{\sigma}_i = 3p$, the curve declines under compression if $\xi = (1 - 2\sqrt{7})/9$, $\varepsilon_i = 0.47695$, $\varepsilon_j = 1.34637$, $J = 0.929825$, and $\hat{\sigma}_i = -0.195797E_0$ and under tension if $\xi = (1 + 2\sqrt{7})/9$, $\varepsilon_i = 3.72429$, $\varepsilon_j = 0.659662$, $J = 1.27304$, and $\hat{\sigma}_i = 0.4787E_0$. Under tension, when $\xi = 2\sqrt{3} - 3$, $\varepsilon_i = 2.21003$, $\varepsilon_j = 0.808927$, $J = 1.20256$, and $\hat{\sigma}_i = 0.385641E_0$, a nonaxisymmetric solution, shown by the dashed curve in Fig. 4, branches off from the axisymmetric solution.

Let us consider the behavior of $\hat{\sigma}_i$ for transverse fibers of constant lengths: $\varepsilon_j = \varepsilon_k = 1$. From (3.4), we obtain $p = 0.5K(1 - \varepsilon_i^{-1})$, $\hat{\sigma}_i = 8\beta\varepsilon_i^{1/2}(\varepsilon_i - 1)(2 + \varepsilon_i)^{-3} + p$, and $\hat{\sigma}_j = \hat{\sigma}_k = (3p - \hat{\sigma}_i)/2$, whose dependences on the parameter $-1 \leqslant \xi = (\varepsilon_i - 1)/(\varepsilon_i + 1) \leqslant 1$ are shown in Fig. 5 (solid and dashed curves, respectively). Under compression, the stresses increase in absolute value without bound; under tension, the value of $\hat{\sigma}_i > 0$ grows, reaches a maximum, and then decreases and $\hat{\sigma}_j$ increases. The deformation of the material is considered stable.

For all the loading cases considered above, ramification of solutions and transition to declining diagrams with unstable strains occurs for $\det D = 0$. For nonunique solutions and solutions with declining diagrams, among the eigenvalues of the matrix $D$ there are negative values. In the regions of stable deformation, the stress–strain diagrams are nearly linear over a wide range of stresses.

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